2021
(Maths 1)

Roll No. [ ] Total No. of Pages : 02

Total No. of Questions : 08

B.Tech. (Agriculture Engineering) / B.Tech. (Automation & Robotics) / B.Tech. (Automobile Engineering) / B.Tech. (Civil Engineering) / B.Tech. (CSE) / B.Tech. (Electrical & Electronics Engineering) / B.Tech. (Electrical Engineering) / B.Tech. (ECE) / B.Tech. (Electronics & Electrical Engineering) / B.Tech. (Mechanical Engineering) / PIT B.Tech CSE / PIT B.Tech ECE (Sem.–1)

MATHEMATICS-I

Subject Code : BTAM-101-18

M.Code : 75353

Date of Examination : 01-02-22

Time : 2 Hrs. Max. Marks : 60

INSTRUCTIONS TO CANDIDATES :

1. Attempt any FIVE question(s), each question carries 12 marks.

1. a) Verify Cauchy's Mean value theorem for the functions $f(x) = x^3 - 3x^2 + 2x$, $g(x) = x^3 - 5x^2 + 6x$ in the interval (0, 0.5). (4)

   b) Discuss the convergence of the integral $\int_0^4 \frac{1}{x(4-x)}\,dx$. (4)

   c) Find the volume of the solid generated by revolving the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2}$ about the x-axis. (4)

2. a) Evaluate $\lim_{x\to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x\sin x}$. (6)

   b) Calculate the approximate value of $\sqrt{10}$ to four decimal places by taking the first four terms of an appropriate Taylor's series. (6)

3. a) Let $f(x, y)$ be a function defined as $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}; & (x,y) \neq (0,0) \\ 0; & (x,y) = (0,0) \end{cases}$.

   Show that $f_x(0, 0)$ and $f_y(0,0)$ exists, although $f(x, y)$ is discontinuous at (0, 0). (6)

   b) A rectangular box, open at the top, is to have a volume of 32 cubic feet. Find the dimensions of the box requiring least material for its construction. (6)

4. a) Find the area bounded by parabolas $y^2 = 4 - x$ and $y^2 = 4 - 4x$. (6)

b) Evaluate $\int_0^\infty \int_0^x x e^{-\frac{x^2}{y}} \, dy dx$ by changing the order of integration. (6)

5. a) Discuss the convergence / divergence of the series : $\sum \left(1 + \frac{1}{\sqrt{n}}\right)^{-\frac{3}{n^2}}$. (6)

b) Discuss the convergence / divergence of the series : (6)

$$x + \frac{2^2 x^2}{\lfloor 2} + \frac{3^3 x^3}{\lfloor 3} + \frac{4^4 x^4}{\lfloor 4} + \frac{5^5 x^5}{\lfloor 5} + \ldots\ldots\infty.$$

6. a) Discuss the convergence / divergence of the series : (7)

$$1 + \frac{\alpha\beta}{1.\gamma} x + \frac{\alpha(\alpha+1)\,\beta(\beta+1)}{1.2.\gamma(\gamma+1)} x^2 + \frac{\alpha(\alpha+1)\,(\alpha+2)\,\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)\,(\gamma+2)} x^2 + \ldots\infty;$$

b) Show that the harmonic series of order p converges for $p > 1$ and diverges for $p \le 1$. (5)

7. a) Compute the inverse of the matrix $\begin{pmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$ by elementary row transformations. (6)

b) For what values of λ and μ do the system of equations $x + y + z = 6$, $x + 2y + 3z = 10$, $x + 2y + \lambda z = \mu$ have (i) No solution (ii) A unique solution. (6)

8. a) Determine the eigen values and corresponding eigen values of the following matrix $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$. (6)

b) If A and B are symmetric matrices of the same order, then show that AB is symmetric if and only if A and B commute. (6)

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B. Tech (CSE) | (EE) | (ECE) (ME)

ECE/ME
Maths-1
2021

Semester-1

Mathematics-I

Subject Code: BTAM101-18

M. Code: 75353

Date of Examination 01-02-22

Q1 (a) Verify Cauchy's Mean Value Theorem for the function $f(x) = x^3 - 3x^2 + 2x$, $g(x) = x^3 - 5x^2 + 6x$ in the interval $(0, 0.5)$.

Soln Cauchy's Mean Value Theorem states that if a function is continuous on close interval $[a,b]$ and differentiable on open interval (a,b) then there exists at least one 'c' in (a,b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Here $f(x) = x^3 - 3x^2 + 2x$ and $g(x) = x^3 - 5x^2 + 6x$

$f'(x) = 3x^2 - 6x + 2$ $\qquad$ $g'(x) = 3x^2 - 10x + 6$

$f(0.5) = (0.5)^3 - 3(0.5)^2 + 2(0.5)$

$\boxed{f(0.5) = 0.375}$

$f(0) = 0$

$g(0.5) = (0.5)^3 - 5(0.5)^2 + 6(0.5)$

$\boxed{g(0.5) = 4.125}$

$g(0) = 0$

$$\frac{f'(c)}{g'(c)} = \frac{f(0.5) - f(0)}{g(0.5) - g(0)} = \frac{0.375 - 0}{4.125 - 0}$$

$$\frac{3c^2 - 6c + 2}{3c^2 - 10c + 6} = \frac{0.375}{4.125} = \frac{375}{4125}$$

On simplifying RHS

$$\frac{3c^2-6c+2}{3c^2-10c+6} = \frac{1}{11}$$

cross multiplying

$$11(3c^2-6c+2) = 3c^2-10c+6$$

$$33c^2-66c+22 = 3c^2-10c+6$$

$$30c^2-56c+16=0$$

$$15c^2-28c+8=0$$

Its roots are given by

$$c = \frac{28 \pm \sqrt{(28)^2-4(15)(8)}}{30}$$

$$= \frac{28 \pm \sqrt{304}}{30}$$

$$= \frac{28 \pm 17.4}{30} \quad \text{so} \quad \frac{28+17.4}{30} = \frac{45.4}{30} = 1.5 \notin (0,0.5)$$

$$\text{and} \quad \frac{28-17.4}{30} = \frac{10.6}{30} = 0.3 \in (0,0.5)$$

So for $\underline{c = 0.3}$ Cauchy's Mean value theorem is verified.

(b) Discuss the Convergence of the integral $\int_0^4 \frac{1}{x(4-x)}\,dx$

Soln We need to Examine the behaviour of this integral in the interval $[0,4]$

This fraction is undefined when denominator is zero, so we need to find value of x for which

$$x(4-x)=0$$

So Eqn is satisfied Either when $\boxed{x=0}$ or $4-x=0$ i.e. $\boxed{x=4}$

Therefore the integrand has vertical asymptotes at $x=0$ and $x=4$.

So we need to check the behaviour of the integrand between these points.

We examine the signs of numerator and denominator in different intervals.

(1) For $0<x<4$

x is positive and $4-x$ is also positive

Hence Denominator $x(4-x)$ is positive.

(2) For $x<0$ and $x>4$, one of x or $4-x$ is negative making $x(4-x)$ negative.

Therefore, the integrand is positive on the interval $0<x<4$ except at the points where it is undefined ($x=0$ and $x=4$).

Since the integrand is continuous on the interval $(0,4)$ except for removable discontinuities at $x=0$ and $x=4$ and it is positive, the integral $\int_0^4 \frac{1}{x(4-x)}\,dx$ converges.

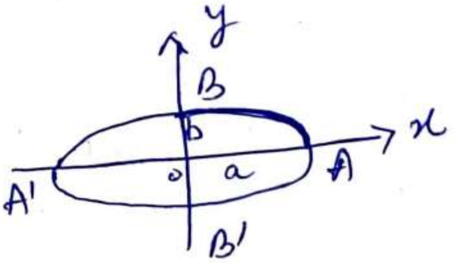
(c) Find the volume of solid generated by revolving the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}$ about x-axis

The given ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\therefore \frac{y^2}{b^2}=1-\frac{x^2}{a^2} \quad \Rightarrow y^2=\frac{b^2}{a^2}(a^2-x^2) \quad (1)$

Required volume $= 2\times$ volume generated by arc BA about x-axis

$$= 2\int_0^a \pi y^2 dx$$



$$= 2\pi \int_0^a \frac{b^2}{a^2}(a^2-x^2)\,dx \quad \text{using (1)}$$

$$= 2\pi \frac{b^2}{a^2} \int_0^a (a^2-x^2)\,dx$$

$$= \frac{2\pi b^2}{a^2}\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$= \frac{2\pi b^2}{a^2}\left[\left(a^3 - \frac{a^3}{3}\right) - (0-0)\right]$$

$$= \frac{2\pi b^2}{a^2} \times \frac{2a^3}{3} = \frac{4\pi a b^2}{3}$$

Q2 (a) Evaluate $\lim_{x\to 0} \dfrac{e^x - e^{-x} - 2\log(1+x)}{x\sin x}$

Soln

$$= \lim_{x\to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x\sin x}$$

$$= \lim_{x\to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x\sin x} \cdot \frac{e^x+e^{-x}}{e^x+e^{-x}} \quad \left(\text{multiplying and dividing by } e^x+e^{-x}\right)$$

$$= \lim_{x\to 0} \frac{e^{2x} - 1 - 2\log(1+x)}{x\sin x\,(e^x+e^{-x})}$$

Here, e^{2x} in numerator approaches 1 as $x \to 0$ and $\log(1+x)$ approaches 0.

$$= \lim_{x\to 0} \frac{1-1-2\log(1+x)}{x\sin x\,(e^x+e^{-x})}$$

$$= \lim_{x\to 0} \frac{-2\log(1+x)}{x\sin x\,(e^x+e^{-x})}$$

As $x \to 0$, $\log(1+x)$ behaves like x and $\sin x$ behaves like x.

$$= \lim_{x \to 0} \frac{-2x}{x^2(e^x+e^{-x})}$$

$$= \lim_{x \to 0} \frac{-2}{x(e^x+e^{-x})}$$

as $x \to 0$ the denominator $(e^x + e^{-x})$ approaches 0. Therefore the limit becomes:

$$= \frac{-2}{0^-}$$

$$= -\infty$$

Hence limit approaches negative ∞

(b) Calculate the approximate value of $\sqrt{10}$ to four decimal places by taking the first four terms of an appropriate Taylor's series.

Soln

Taylor Series: $f(a+h) = f(a) + h \cdot f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3} f'''(a) + \ldots$

we can write $10 = 9+1$

using Taylor series method

$$f(10) = f(9+1)$$

$$\Rightarrow a = 9, \quad h = 1$$

$$f(a) = a^{1/2} = f(a+1) = \sqrt{10}$$

Substituting the value of 'a' we have

$$f(a) = 9^{1/2} = 3$$

$$f'(a) = \frac{1}{2} a^{1/2 - 1} = \frac{1}{2} a^{-1/2} = \frac{1/2}{a^{1/2}} = \frac{1}{2a^{1/2}}$$

Substituting value of 'a'

$$f'(a) = \frac{1}{2(9)^{1/2}} = 0.166 \ldots$$

$$f''(a) = \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) a^{-1/2 - 1} = -\frac{1}{4} a^{-3/2} = \frac{-1/4}{a^{3/2}} = \frac{-1}{4a^{3/2}}$$

Substituting value of 'a' we have

$$f''(a) = -\frac{1}{4(9)^{3/2}} = 0.00925$$

$$f'''(a) = \left(-\tfrac{3}{2}\right)\left(-\tfrac{1}{4}\right) a^{-3/2-1} = \frac{3}{8} a^{-5/2} = \frac{\frac{3}{8}}{a^{5/2}} = \frac{3}{8a^{5/2}}$$

Put value of 'a'

$$f'''(a) = \frac{3}{8(9)^{5/2}} = 0.00514$$

From Taylor series method, we have

$$\sqrt{10} = f(a+1) = f(a) + 1.f'(a) + \frac{1^2}{2!}f''(a) + \frac{1^3}{3!}f'''(a) + \cdots$$

$$\sqrt{10} = f(a+1) = f(a) + 1.f'(a) + \frac{1}{2!}f''(a) + \frac{1}{3!}f'''(a) + \cdots$$

$f(a) = 3$, $f'(a) = 0.166$, $f''(a) = 0.00925$

$f'''(a) = 0.000514$

$$\Rightarrow \sqrt{10} = f(a+1) = 5 + 0.1 + \frac{0.00925}{2} + \frac{0.000514}{3!}$$

$$\underline{\sqrt{10} \approx 5.10471}$$

$\boxed{\sqrt{10} = 5.1047}$ correct to four decimal places

Q3 (a) Let $f(x,y)$ be a function defined as

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & ; \ (x,y) \neq (0,0) \\ 0 & ; \ (x,y) = (0,0) \end{cases}$$

Show that $f_x(0,0)$ and $f_y(0,0)$ Exists, although $f(x,y)$ is discontinuous at $(0,0)$

Soln we find $f_x(x,y)$ and $f_y(x,y)$

$$f_x(x,y) = \frac{\partial f}{\partial x} = \frac{y(x^2+y^2) - xy(2x)}{(x^2+y^2)^2}$$

$$f_y(x,y) = \frac{\partial f}{\partial y} = \frac{x(x^2+y^2) - xy(2y)}{(x^2+y^2)^2}$$

At $(0,0)$

$$f_x(x,y) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \underset{h\to 0}{lt} \frac{0}{h} = 0$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(0,0+k) - f(0,0)}{k} = \underset{k\to 0}{lt} \frac{0}{k} = 0$$

$\therefore$ both $f_x(0,0)$ and $f_y(0,0)$ Exist.

now let $f(x,y) \to (0,0)$ along the curve $y = mx^2$

$$\underset{(x,y)\to(0,0)}{lt} f(x,y) = \underset{(x,y)\to(0,0)}{lt} \frac{xy}{x^2+y^2}$$

$$= \underset{x\to 0}{lt} \frac{x\cdot(mx^2)}{x^2+m^2x^4}$$

$$= \underset{x\to 0}{lt} \frac{mx^3}{x^2+m^2x^4}$$

$$= \underset{x\to 0}{lt} \frac{m}{1/x + m^2x}$$ which is not unique as it depends upon m.

$\therefore \underset{(x\to y)\to(0,0)}{lt} f(x,y)$ does not Exist

$\therefore$ f is discontinuous at $(0,0)$

(b) A rectangular box, open at the top, is to have volume of 32 cubic feet. Find the dimensions of the box requiring least material for its construction.

Soln Let x, y, z be the edges of the open box and S be its surface.

$$S = xy + 2yz + 2zx \quad (1)$$

Volume = 32 cubic ft

$$\therefore \quad xyz = 32$$

$$z = \frac{32}{xy}$$

$\therefore$ from (1) $\quad S = xy + 2y \cdot \frac{32}{xy} + 2x \cdot \frac{32}{xy}$

$$S = xy + \frac{64}{x} + \frac{64}{y}$$

Step1 $\quad \frac{\partial S}{\partial x} = y - \frac{64}{x^2}, \quad \frac{\partial S}{\partial y} = x - \frac{64}{y^2}$

Step2 Let solve $\frac{\partial S}{\partial x} = 0$ and $\frac{\partial S}{\partial y} = 0$

$$y - \frac{64}{x^2} = 0 \quad (2)$$

and $\quad x - \frac{64}{y^2} = 0 \quad (3)$

from (2) $\quad y = \frac{64}{x^2}$

from (3) $\quad \cancel{y = \frac{64}{x^2}} \; x - \frac{64}{(64)^2} \times x^4 = 0$

or $\quad (64x - x^4) = 0$

$$x(64 - x^3) = 0$$

$$x = 0, 4$$

Now $x = 0$ does not give any finite value of y

when $x = 4 \quad y = \frac{64}{16} = 4 \quad \therefore$ critical point is $(4, 4)$

Step III : $A = \frac{\partial^2 S}{\partial x^2} = \frac{128}{x^3}$, $B = \frac{\partial^2 S}{\partial x \partial y} = 1$, $C = \frac{\partial^2 S}{\partial y^2} = \frac{128}{y^3}$

$AC - B^2 = \left(\frac{128}{x^3 y^3}\right)^2 - 1$

Step IV : At $(4, 4)$

$AC - B^2 = \frac{(128)^2}{64 \times 64} - 1 = 4 - 1 = 3 > 0$

$A = \frac{128}{(4)^3} = 2 > 0$

S is least when $x = 4$, $y = 4$

$\therefore$ from (2) $\quad z = \frac{32}{4 \times 4} = 2$

$\therefore$ dimensions of the box requiring least material are 4m, 4m and 2m.

Q4 (a) Find area bounded by parabola's $y^2 = 4 - x$ and $y^2 = 4 - 4x$

Soln To find the area bounded by the parabola's $y^2 = 4 - x$ and $y^2 = 4 - 4x$ we need to find point of intersection.

So, Equating both $\quad 4 - x = 4 - 4x$

$-x = -4x$

$3x = 0$

$\boxed{x = 0}$

Put in $y^2 = 4 - x$

$y^2 = 4$

$\boxed{y = \pm 2}$

So the points of intersection are $(0, 2)$ and $(0, -2)$

The Area A can be written as:

$$A = \int_0^2 |(4-x)-(4-4x)|\, dx$$

$$A = \int_0^2 |3x|\, dx$$

$$A = \int_0^2 3x\, dx$$

$$A = \frac{3x^2}{2}\Big|_0^2$$

Evaluate the Expression:

$$A = \frac{3}{2}(2)^2 - \frac{3}{2}(0)^2$$

$$A = \frac{3}{2} \times 4 = 6$$

So area bounded by parabolas $y^2 = 4-x$ and $y^2 = 4-4x$ is 6.

(b) Evaluate $\int_0^{\infty}\int_0^{x} x\, e^{-\frac{x^2}{y}}\, dy\, dx$ by changing the order of integration.

Soln

The given integral is $\int_0^{\infty}\int_0^{x} x e^{-x^2/y}\, dy\, dx$, By Changing the order. The integral becomes:

$$\int_0^{x}\int_{y}^{\infty} x e^{-x^2/y}\, dx\, dy$$

Integrate with respect to x first

$$\int_0^{\infty} \quad \int_0^{x} x\, e^{-\frac{x^2}{y}}\, dx$$

To solve this, lets make a substitution

Let $u = \frac{x^2}{y}$ and $du = -\frac{x^2}{y^2}\, dy$

The limit of integration become:

When $x=0$, $u=0$

$x=x$, $u=\frac{x^2}{y}$.

Now the integral can be written as:

$$= -\frac{1}{2}\int_0^x e^u \, du$$

Integrate wrt 'u':

$$= \frac{-1}{2}\int_0^x e^u \, du$$

$$= \frac{-1}{2}\left[e^u\right]_0^x$$

$$= \frac{-1}{2}(e^x - 1)$$

So the new Expression for double Integral is

$$\int_0^\infty \left(\frac{-1}{2}(e^x-1)\right) dy$$

Integrate with respect to y from 0 to ∞:

$$\int_0^\infty \left(\frac{-1}{2}(e^x-1)dy\right)$$

$$= \left(\frac{-1}{2}(e^x-1)\right)\Big|_0^\infty$$

Evaluate the limits:

$$\lim_{y\to\infty}\left(\frac{-1}{2}(e^x-1)\right) - \left(\frac{-1}{2}(e^0-1)\right)$$

$$= \lim_{y\to\infty} \frac{-1}{2}(e^x-1) + 1/2$$

Since e^x approaches ∞ as $x\to\infty$ therefore the limit is not defined and the given Integral is divergent.

Q5 (a) Discuss the Convergence / divergence of the series:

$$\sum \left(1+\frac{1}{\sqrt{n}}\right)^{-3/n^2}.$$

Soln:

To check the Convergence or divergence of the series $\sum\left(1+\frac{1}{\sqrt{n}}\right)^{-3/n^2}$, we use limit Comparison Test.

Let $a_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-3/n^2}$

and, $b_n = n^{-3/2}$

So
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{\sqrt{n}}\right)^{-3/n^2}}{n^{-3/2}}$$

we know that $\frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$

so the above Expression becomes

$$\approx \lim_{n\to\infty} \frac{(1)^{-3/n^2}}{n^{-3/2}}$$

Also $\lim_{n\to\infty} n^{3/2} = \infty$

Since the limit is infinite, the series $\sum_{n=1}^{\infty} b_n$ diverges.

$\therefore$ By limit Comparison Test if $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and b_n is positive and Convergent by p Test $(3/2 > 1)$ then $\sum a_n$ also diverges

$$\therefore \sum\left(1+\frac{1}{\sqrt{n}}\right)^{-3/n^2} \text{ diverges.}$$

(b) Discuss the Convergence / divergence of the series:

$$x + \frac{2^2x^2}{\lfloor 2} + \frac{3^3x^3}{\lfloor 3} + \frac{4^4x^4}{\lfloor 4} + \frac{5^5x^5}{\lfloor 5} + \dots \infty$$

Soln

The given series is Σa_n where $a_n = \frac{n^n x^n}{\lfloor n}$

$\therefore a_{n+1} = \frac{(n+1)^{n+1}}{\lfloor n+1} x^{n+1}$

$$\therefore \frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1} x^{n+1}}{\lfloor n+1} \times \frac{\lfloor n}{n^n x^n}$$

$$= \frac{(n+1)^{n+1}}{(n+1)\lfloor n} x \frac{\lfloor n}{n^n}$$

$$= \frac{(n+1)^n}{n^n} x$$

$$= \left(\frac{n+1}{n}\right)^n x = \left(1+\frac{1}{n}\right)^n x$$

$$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = e \cdot x$$

$\therefore$ By Ratio Test Σa_n Cgs for $ex < 1$

ie $x < \frac{1}{e}$ and diverges for $ex > 1$; $x > \frac{1}{e}$

for $x = \frac{1}{e}$, Ratio test fails.

when $x = \frac{1}{e}$ $\quad \frac{a_n}{a_{n+1}} = e\left(1+\frac{1}{n}\right)^{-n}$

$$\therefore \log \frac{a_n}{a_{n+1}} = \log e + \log\left(1+\frac{1}{n}\right)^{-n}$$

$$= 1 - n \log\left(1+\frac{1}{n}\right)$$

$$= 1 - n\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^2} - \dots\right]$$

$$\log \frac{a_n}{a_{n+1}} = \frac{1}{2n} - \frac{1}{3n^2} + \dots$$

$$n \log \frac{a_n}{a_{n+1}} = \frac{1}{2} - \frac{1}{3n} + \dots$$

$\lim_{n\to\infty} n \log \frac{a_n}{a_{n+1}} = \frac{1}{2} < 1$, $\therefore$ So By log Test Σa_n diverges

Q6 (a) Discuss the Convergence/Divergence of the series:

$$1 + \frac{\alpha . \beta}{1 . \gamma} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3\gamma(\gamma+1)(\gamma+2)}x^3 + \dots \infty$$

Soln

For the given series (after neglecting 1st Term)

$$a_n = \frac{\alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1)\beta(\beta+1)(\beta+2)\dots(\beta+n-1)}{1.2.3.\dots n.\gamma(\gamma+1)(\gamma+2)\dots\gamma(\gamma+n-1)}x^n$$

$$a_{n+1} = \frac{\alpha(\alpha+1)\dots(\alpha+n-1)(\alpha+n)\cdot\beta(\beta+1)\dots(\beta+n-1)(\beta+n)}{1.2.3\dots n(n+1)\gamma(\gamma+1)\dots(\gamma+n-1)(\gamma+n)}x^{n+1}$$

$$\frac{a_{n+1}}{a_n} = \frac{(\alpha+n)(\beta+n)}{(n+1)(\gamma+n)}x = \frac{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\gamma/n\right)}$$

$$\underset{n\to\infty}{Lt} \frac{a_{n+1}}{a_n} = \frac{(1+0)(1+0)}{(1+0)(1+0)}x = x$$

$\therefore$ By ratio test, Σa_n Converges for $x<1$ and diverges for $x>1$

For $x=1$, Ratio Test fails.

so at $x=1$,

$$\frac{a_n}{a_{n+1}} = \frac{(n+1)(\gamma+n)}{(n+\alpha)(n+\beta)} = \frac{\left(1+1/n\right)\left(1+\frac{\gamma}{n}\right)}{\left(1+\frac{\alpha}{n}\right)\left(1+\beta/n\right)}$$

$$= \left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)\left(1+\frac{\alpha}{n}\right)^{-1}\left(1+\beta/n\right)^{-1}$$

$$= \left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)\left\{1-\frac{\alpha}{n}+o\left(\frac{1}{n^2}\right)\right\}\left\{1-\frac{\beta}{n}+o\left(\frac{1}{n^2}\right)\right\}$$

$$= 1 + \frac{1+\gamma-\alpha-\beta}{n} + o\left(\frac{1}{n^2}\right) = 1 + \frac{\mu}{n} + o\left(\frac{1}{n^2}\right)$$

where $\mu = 1+\gamma-\alpha-\beta$

$\therefore$ by Gauss Test, Σa_n Converges for $1+\gamma-\alpha-\beta>1$

ie $\gamma > \alpha+\beta$

and diverges for $1+\gamma-\alpha-\beta < 1$

ie $\gamma < \alpha+\beta$.

(b) Show that the harmonic series of order 'p' Converges for $p>1$ and diverges for $p \leq 1$.

Soln The harmonic series of order 'p' is given by:

$$H_p = \sum_{n=1}^{\infty} \frac{1}{n^p}$$

we use p-series test to show the Convergence and divergence for $p>1$ and $p \leq 1$ respectively.

For $p>1$:

Consider the integral

$$\int_1^{\infty} \frac{1}{x^p}\, dx$$

This integral Converges if $p>1$ and diverges if $p \leq 1$.

The harmonic series H_p can be thought of as the sum of the areas of rectangles under the curve $\frac{1}{x^p}$ from 1 to ∞.

If $p>1$, the integral Converges, which means the area under the curve is finite and therefore the sum of areas of rectangles (the harmonic series) is finite. Thus H_p Converges for $p>1$.

for $p \leq 1$: The integral diverges, indicating that the area under the curve is infinite. Consequently, the sum of areas of rectangles (harmonic series) is infinite. Thus, H_p diverges for $p \leq 1$.

Q7(a) Compute the inverse of the matrix by elementary row transformations $\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

Given matrix $A = \begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

Now $A = IA$

$$\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 8 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$$

$R_2 \to R_2 - 2R_1, \; R_3 \to R_3 - 8R_1$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 0 & -12 & -5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & -8 \end{bmatrix} A$$

$R_3 \to R_3 - 4R_2$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -4 & 0 \end{bmatrix} A$$

$R_3 \to (-)R_3$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ -1 & 4 & 0 \end{bmatrix} A$$

$R_1 \to R_1 + R_2$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & -3 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & -2 \\ -1 & 4 & 0 \end{bmatrix} A$$

$R_2 \to R_2 + R_3$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 5 & -2 \\ -1 & 4 & 0 \end{bmatrix} A$$

$$R_2 \to \frac{R_2}{(-3)}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 1/3 & -5/3 & 2/3 \\ -1 & 4 & 0 \end{bmatrix} A$$

$$R_1 \to R_1 + R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & -\frac{2}{3} & 1/3 \\ 1/3 & -5/3 & \frac{2}{3} \\ -1 & 4 & 0 \end{bmatrix} A$$

$$I = A^{-1} A$$

So, $A^{-1} = \begin{bmatrix} 1/3 & -2/3 & 1/3 \\ 1/3 & -5/3 & 2/3 \\ -1 & 4 & 0 \end{bmatrix}$

(b) For what values of 'λ' and 'μ' do the system of equations $x+y+z=6$, $x+2y+3z=10$, $x+2y+\lambda z=\mu$ have (i) no solution (ii) a unique solution.

Soln The Equations are:

$$x+y+z=6$$
$$x+2y+3z=10$$
$$x+2y+\lambda z=\mu$$

which can be written as: $AX=B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}$ and $B = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$

(ii) The given equations will give a unique solution if $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{vmatrix} \neq 0$ ie, if $\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & \lambda-1 \end{vmatrix} \neq 0$

if $\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & \lambda-3 \end{vmatrix} \neq 0$

ie $\lambda - 3 \neq 0$ ie if $\lambda \neq 3$

$\therefore$ if $\lambda \neq 3$, the given set of Equations has a unique solution no matter what is the value of λ.

(i) for no solution

Take $\lambda = 3$.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \text{ by } R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ by } R_3 \to R_3 - R_2$$

$\therefore \ \rho(A) = 2$

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & 3 & : & \mu \end{bmatrix}$$

rank of $[A:B]$ is 3 if $\mu \neq 10$

$\therefore$ rank of $[A:B]$ and A are not equal if $\mu \neq 10$

$\therefore$ If $\lambda = 3$ and $\mu \neq 10$ then Eqns does not have any solution

Q8 (a) Determine the Eigen values and corresponding Eigen vectors of the following matrix

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Soln

Let $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

$$\lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix}$$

Characteristic equation of matrix A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (8-\lambda)\{(7-\lambda)(3-\lambda)-16\} + 6\{-6(3-\lambda)+8\} + 2\{24-2(7-\lambda)\} = 0$

$\Rightarrow (8-\lambda)\{\lambda^2 - 10\lambda + 5\} + 6\{6\lambda - 10\} + 2\{2\lambda + 10\} = 0$

$\Rightarrow -\lambda^3 + 18\lambda^2 - 85\lambda + 40 + 36\lambda - 60 + 4\lambda + 20 = 0$

$\Rightarrow -\lambda^3 + 18\lambda^2 - 45\lambda = 0$

$\Rightarrow \lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\lambda(\lambda^2 - 18\lambda + 45) = 0$

$\lambda(\lambda - 3)(\lambda - 15) = 0$

$\lambda = 0,\ \lambda = 3,\ \lambda = 15$ are characteristic roots of A

The Eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq 0$ corresponding to Eigen value $\underline{\lambda = 0}$ is given by $AX = 0X$

$$(A-0)X=0$$

$$AX=0$$

$$\Rightarrow \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & -4 & 3 \\ -6 & 7 & -4 \\ 8 & -6 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{by } R_1 \to R_3$$

$$\Rightarrow \begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 10 & -10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{by } R_2 \to R_2+3R_1,\ R_3 \to R_3-4R_1$$

$$\Rightarrow \begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{by } R_3 \to R_3+2R_2$$

The coefficient matrix of these equations is of rank 2.
So these equations have only $3-2=1$ LI solution.
The equations can be written as

$$2x-4y+3z=0$$
$$-5y+5z=0 \quad \Rightarrow \boxed{y=z}$$

Take $\boxed{y=z=1}$

$$\therefore\ 2x-4+3=0 \qquad \boxed{x=1/2}$$

$X = \begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix}$ is eigen vector of A for $\lambda=0$

<u>for $\lambda=3$</u> Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ to be Eigen vector

$$AX=3X \quad \text{or} \quad (A-3I)X=0$$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -2 & -2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ by } R_1 \to R_1 + R_2$$

$$\begin{bmatrix} -1 & -2 & -2 \\ 0 & 16 & 8 \\ 0 & -8 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ by } R_2 \to R_2 - 6R_1, \; R_3 \to R_3 + 2R_1$$

$$\begin{bmatrix} -1 & -2 & -2 \\ 0 & 16 & 8 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ by } R_3 \to R_3 + \tfrac{1}{2}R_2$$

The coeff matrix of given Equations is of rank 2.
so these Eqn's have $3-2 = 1$ L.I. solution

$$-x - 2y - 2z = 0$$
$$16y + 8z = 0$$
$$\Rightarrow y = -\tfrac{1}{2}z$$

Take $z = 4$ so $y = -2$

and $x = -4$

$X = \begin{bmatrix} -4 \\ -2 \\ 4 \end{bmatrix}$ is eigen vector of A Corresponding to eigen value $\lambda = 3$

for $\lambda = 15$

The Eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq 0$ Corresponding to given value $\lambda = 15$ is given by $AX = 15X$

or $(A - 15I)X = 0$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 & 6 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ by } R_1 \to R_1 - R_2$$

or

$$\begin{bmatrix} -1 & 2 & 6 \\ 0 & -20 & -40 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ by } \begin{array}{l} R_2 \to R_2 - 6R_1 \\ R_3 \to R_3 + 2R_1 \end{array}$$

The coefficient matrix is of rank 2.
So Eqn's have $3-2 = 1$ L.I solution

$$-x + 2y + 6z = 0$$

$$-20y - 40z = 0 \Rightarrow y = -2z$$

Take $\underline{z = 1}$ so $\underline{y = -2}$

$\therefore$ $\underline{x = 2}$

$\therefore$ $X = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ is an eigen vector of A for $\lambda = 15$

(b) If A and B are symmetric matrices of same order, then show that AB is symmetric iff A and B commute.

Soln Forward Direction:

Assume that A and B commute, i.e $AB = BA$

Now we need to show AB is symmetric.

The transpose of a product of matrices is given by the reverse order of the product's transposes

$$(AB)^T = B^T A^T$$

Since A and B are symmetric

$$A^T = A \text{ and } B^T = B$$

So $(AB)^T = BA$

But we know $AB = BA$

So $(AB)^T = AB$

which implies AB is symmetric

Hence proved

Reverse Direction

Conversely, Assume AB is symmetric

Now we want to show that A and B commute

ie $AB = BA$

Let's consider AB being symmetric

$$(AB)^T = AB$$

But we need know that

$$(AB)^T = B^T A^T$$

and since AB is symmetric,

$$B^T A^T = AB.$$

So we have $AB = B^T A^T$

Now, if we take transpose of both sides, we get

$$(AB)^T = (B^T A^T)^T$$

$$AB = (A^T)^T (B^T)^T$$

$$AB = BA$$

Therefore if AB is symmetric then A and B commute

This shows that if A and B are symmetric matrices of same order, then AB is symmetric iff A and B commute.